SOME EXAMPLES OF
COMPLEMENTED MODULAR LATTICES

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(received November 13, 1961)

Let $L$ be a complemented, $\kappa$-complete modular lattice. A theorem of Amemiya and Halperin (see [1], Theorem 4.3) asserts that if the intervals $[0, a]$ and $[0, b]$, $a, b \in L$, are upper $\kappa$-continuous then $[0, a \lor b]$ is also upper $\kappa$-continuous. Roughly speaking, in $L$ upper $\kappa$-continuity is additive. The following question arises naturally: is $\kappa$-completeness an additive property of complemented modular lattices? It follows from Corollary 1 to Theorem 1 below that the answer to this question is in the negative.

A complemented modular lattice is called a Von Neumann geometry or continuous geometry if it is complete and continuous. In particular a complete Boolean algebra is a Von Neumann geometry. In any case in a Von Neumann geometry the set of elements which possess a unique complement form a complete Boolean algebra. This Boolean algebra is called the centre of the Von Neumann geometry. Theorem 2 shows that any complete Boolean algebra can be the centre of a Von Neumann geometry with a homogeneous basis of order $n$ (see [3] Part II, definition 3.2 for the definition of a homogeneous basis), $n$ being any fixed natural integer.

Preliminaries

We first recall some properties of regular rings. The definitions and proofs can be found in [3] part II, Chap. II or [2], §3. We always assume that the regular ring has a unit element which will be denoted by 1.

If $S$ is a regular ring, $\text{L}_S$ ($\text{R}_S$) denotes the
complemented modular lattices of principal left (right) ideals.
The mapping which takes each element of $\text{L}_S$ into its right
annihilator is a dual-isomorphism of $\text{L}_S$ onto $\text{R}_S$. Under
this map the principal left ideal $(e)_L$ generated by the
idempotent $e$ goes into the principal right ideal $(1-e)_R$.

If $S$ is a regular ring, the ring $S_n$ of $n \times n$ matrices
with entries in $S$ is also regular. There exists a lattice
isomorphism between $\text{L}_S$ ($\text{R}_S$) and the lattice of finitely
generated submodules of the left (right) $S$-module of $n$-tuples
$(a_1, a_2, \ldots, a_n)$, $a_1 \in S$. Since $S$ is regular, for every
$A \in S_n$ there exists an idempotent matrix $E$ such that
$(E)_L = (A)_L$. Moreover, it is possible to choose

$$
E = \begin{pmatrix}
e_1 & 0 & \cdots & 0 \\
c_{21} & e_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
c_{n1} & c_{n2} & \cdots & e_n
\end{pmatrix}
$$

where $e_i = e_i$, $c_{ij} = c_{ij}$, $e_i c_j = 0$, for $i, j = 1, 2, \ldots, n$
and $c_{ij} = 0$, for $j > i$. Such a matrix is called a left
canonical matrix. An idempotent matrix such that

$$
e_i^2 = e_i, c_{ij} e_i = c_{ij}, e_i c_{ij} = 0 \text{ for } i, j = 1, 2, \ldots, n\text{ and}
c_{ij} = 0 \text{ for } j > i$$

is called right canonical. For every $A \in S_n$
there exists a right canonical matrix $E$ such that $(A)_R = (E)_R$.

Notice that if $E$ is a right (left) canonical matrix then $1-E$
is left (right) canonical.

In what follows our regular ring $S$ will be the Boolean
ring $B$ defined by a Boolean algebra $\mathcal{L}$, that is, the elements
of $B$ are those of $\mathcal{B}$ and

$$a + b = ab' \cup ba', \quad ab = a \wedge b,$$

where $c'$ denotes the complement of $c \in \mathcal{B}$. The notation $c = a \cup b$ implies that $ab = 0$. If $\mathcal{I}$ is an ideal of $\mathcal{L}$, it defines an ideal $I$ of $B$. There exists a 1-1 correspondence between the elements of $\mathcal{L}$ and the principal ideal of $B$.

In the ring $S_n$, there is in general more than one left (right) canonical matrix corresponding to an element $A \in S_n$. However, if two left canonical matrices $E$ and $F$ are such that $(E)_{\mathcal{I}} = (F)_{\mathcal{I}}$ and they have the same idempotents down the main diagonal, then $E = F$. This follows from the fact that $EF = E$ if $(E)_{\mathcal{I}} = (F)_{\mathcal{I}}$. Although in general the element $e_1$ is not uniquely defined by $A$, the ideal $(e_1)_{\mathcal{I}}$ is unique.

Since in the Boolean ring $B$ any principal ideal is defined by a unique element, any principal left ideal of $B_n$ is defined by a unique left canonical matrix. We will identify the elements of $L_B$ with the corresponding left canonical matrices.

Some examples of complemented modular lattices

Throughout this section $\mathcal{L}$ will be a Boolean algebra, $\mathcal{I}$ an ideal of $\mathcal{L}$, and $B$ and $I$ the corresponding Boolean ring and ideal. $\mathcal{J}$ denotes the cardinal power of the set $J$.

**THEOREM 1.** Let $L$ consist of the $2 \times 2$ left canonical matrices

$$A = \begin{pmatrix} e_1 & 0 \\ a & e_2 \end{pmatrix}, \quad \text{where } e_1, e_2 \in B \text{ and } a \in I. \quad \text{For } A_1, A_2 \in L, \text{ define } A_1 \leq A_2 \text{ if } (A_1)_{\mathcal{I}} \subseteq (A_2)_{\mathcal{I}} \text{ where } (A)_{\mathcal{I}} \text{ is the principal left ideal of } B_2 \text{ generated by } A. \text{ Then } L \text{ is a complemented modular lattice. Moreover, the following conditions are equivalent}
(i) \( L \) is an \( \mathcal{F}_a \)-complete \( \mathcal{F}_a \)-sublattice of \( \overline{L}_{B_2} \).

(ii) \( L \) is an \( \mathcal{F}_a \)-complete \( \mathcal{F}_a \)-continuous \( \mathcal{F}_a \)-sublattice of \( \overline{L}_{B_2} \).

(iii) \( \mathcal{F} \) is an \( \mathcal{F}_a \)-ideal and \( \mathcal{B} \) is \( \mathcal{F}_a \)-complete.

Proof. Let \( R \) be the set of right canonical matrices

\[
A = \begin{pmatrix}
e_1 & 0 \\ a & e_2 \end{pmatrix}, \quad e_1, e_2 \in B \quad \text{with} \quad a \in I,
\]

ordered by the relation

\[ A_1 \leq A_2 \text{ if } (A_1)_r \subseteq (A_2)_r. \]

Then the dual isomorphism between \( \overline{L}_{B_2} \) and \( \overline{R}_{B_2} \) induces a dual isomorphism between \( L \) and \( R \). Hence any statement about \( L \) implies its dual, since what we prove for \( L \) can be proved as well for \( R \).

We show first that \( L \) is a complemented modular lattice.

When \( \mathcal{F} = \mathcal{B} \) the ordered set defined in the theorem coincides with \( \overline{L}_{B_2} \) and there is nothing to prove. When \( \mathcal{F} \neq \mathcal{B} \) we will prove that \( L \) is a sublattice of \( \overline{L}_{B_2} \). For this we use the lattice isomorphism between the principal left ideals of \( B_2 \) and the finitely generated submodules of the left \( B \)-module of 2-tuples \( (a_1, a_2) \), \( a_i \in B \). If \( \{(a_1, a_2)\} \) denotes the left submodule generated by the vector \( (a_1, a_2) \) then the module \( M \) corresponding to the canonical matrix

\[
\begin{pmatrix}
e_1 & 0 \\ a & e_2 \end{pmatrix}
\]

has the form

(1) \( M = \{(e_1, 0)\} \oplus \{(a, e_2)\} = \{(e_1, 0)\} \oplus \{(a, a)\} \oplus \{(0, a_0)\} \)

where \( a_0 = e_2 a^t \) and \( \oplus \) indicates direct sum. Since the matrix is canonical \( e_2 = a \cup a_0 \).
It is clear that the only elements of \( M \) whose second or first component is zero are the elements of the submodules \( \{(e_{1}^{0}, 0)\} \) or \( \{(0, a_{o})\} \), respectively. The elements of \( M \) of the form \((c, c)\) are the elements of \( \{(a \cup_{1} a_{o}, a \cup_{1} a_{o})\} \).

The module

\[
(2) \quad N = \{(f_{1}, 0)\} \oplus \{(b, b)\} \oplus \{(0, b_{o})\},
\]

where \( b \in I, bf_{1} = bb_{o} = 0 \), corresponds to the canonical matrix

\[
\begin{pmatrix}
 f_{1} & 0 \\
 b & f_{2}
\end{pmatrix},
\]

where \( f_{2} = b \cup b_{o} \). Now \( N \) contains \( M \) if and only if

\[
e_{1} \leq f_{1}, \quad a_{o} \leq b_{o} \quad \text{and} \quad a \leq b \cup f_{1} b_{o},
\]
or what is equivalent,

\[
e_{1} \leq f_{1}, \quad e_{2} \leq f_{2}, \quad a \leq b \cup f_{1} f_{2} \quad \text{and} \quad a_{o} b = 0.
\]

In general given two modules \( M \) and \( N \) defined by

(1) and (2)

\[
M \cap N = \{(e_{1} \cup f_{1}, 0)\} + \{(a \cup b, a \cup b)\} + \{(0, a_{o} \cup b_{o})\} = \{(e_{1} \cup f_{1} \cup b_{a_{o}} \cup b_{a_{o}} a, 0)\} \oplus \{(c, c)\} \oplus \{(0, a_{o} \cup b_{o} \cup b_{e_{1} \cup af_{1}})\}
\]

where \( c = af_{1} b_{1} \cup e_{1} a_{1} b_{1} \leq a \cup b \in I \). Hence \( M \cup N \in L \). By duality \( M \cap N \in L \). Therefore \( L \) is a sublattice of a modular lattice and is itself modular. Since

\[
M' = \{(e_{1} a_{1}', 0)\} \oplus \{(0, a_{o}')\}
\]
is a complement of the module \( M \), \( L \) is a complemented modular lattice.

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Our next step is to show that if \( \mathcal{V} \) is \( \mathcal{F} \)-complete then \( \overline{L}_{B_2} \) is \( \mathcal{F} \)-complete. It is sufficient to show that \( \overline{L}_{B_2} \) is upper \( \mathcal{F} \)-complete, because the lower \( \mathcal{F} \)-completeness follows by duality.

Let \( A^\beta = \begin{pmatrix} e_1^\beta & 0 \\ a_1^\beta & e_2^\beta \end{pmatrix} \in \overline{L}_{B_2} \) for all \( \beta \in J \),

where \( J \leq \mathcal{F} \). It is immediate that if \( \mathcal{V} \) is \( \mathcal{F} \)-complete, the union of the corresponding modules

\[
M_3 = \{(e_1^\beta, 0)\} \oplus \{(a_1^\beta, a_2^\beta)\} \oplus \{(0, a_o^\beta)\} \quad \text{where} \quad a_o^\beta = e_2^\beta (a_1^\beta)'
\]

is the module

\[
M = \{ (\cup e_1^\beta, 0) \} + \{ (\cup a_1^\beta, \cup a_2^\beta) \} + \{ (0, \cup a_o^\beta) \}
\]

which corresponds to the canonical matrix

\[
A = \begin{pmatrix}
\cup e_1^\beta & (\cup a_1^\beta) & (\cup a_o^\beta) & 0 \\
0 & d & (\cup a_1^\beta) & (\cup a_o^\beta)
\end{pmatrix}
\]

where 

\[
d = (\cup a_1^\beta) \cdot (\cup e_1^\beta \cup ((\cup a_1^\beta) \cdot (\cup a_o^\beta))').
\]

Now we are ready to prove the equivalence of conditions (i), (ii), (iii).

(i) implies (ii). This is a consequence of the additivity of upper \( \mathcal{F} \)-continuity in complemented \( \mathcal{F} \)-complete modular lattices. For, if

\[
X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

the intervals \([0, X]\) and \([0, Y]\) are both isomorphic to \( \mathcal{V} \);
hence \( L = [0, X \cup Y] \) is upper \( \mathcal{S} \)-continuous. Using duality we get that \( L \) is \( \mathcal{S} \)-continuous.

(ii) implies (iii). Since \( \mathcal{S} \) is isomorphic to the interval \([0, X]\), if \( L \) is \( \mathcal{S}_\alpha \)-complete then \( B \) is \( \mathcal{S}_\alpha \)-complete.

Now let

\[
C^\beta = \begin{pmatrix} 0 & 0 \\ a^\beta & a^\beta \end{pmatrix} \in L
\]

for all \( \beta \in J \) and \( J \leq \mathcal{S}_\alpha \). Then

\[
\cup C^\beta = \begin{pmatrix} 0 & 0 \\ \cup a^\beta & \cup a^\beta \end{pmatrix} \in L,
\]

which implies that \( \cup a^\beta \in \mathcal{S} \) and therefore \( \mathcal{S} \) is \( \mathcal{S}_\alpha \)-complete.

(iii) implies (i). Let

\[
A^\beta = \begin{pmatrix} e^\beta_1 & 0 \\ a^\beta & e^\beta_2 \end{pmatrix} \in L \quad \text{for all } \beta \in J,
\]

and \( J \leq \mathcal{S}_\alpha \). Then (3) implies that \( \cup A^\beta \in L \), hence (i) holds.

**COROLLARY 1.** Let \( L \) be as in Theorem 1. Suppose \( \mathcal{S} \) is complete and \( \mathcal{S} \) is an \( \mathcal{S}_\alpha \)-ideal which is not an \( \mathcal{S}_{\alpha+1} \)-ideal. Then

(a) \( L \) contains two elements \( X \) and \( Y \) such that the intervals \([0, X]\) and \([0, Y]\) are complete and continuous and \( L = [0, X \cup Y] \).

(b) \( L \) is \( \mathcal{S}_\alpha \)-complete and \( \mathcal{S}_\alpha \)-continuous but not \( \mathcal{S}_{\alpha+1} \)-complete.

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Proof. The only thing which has to be proved is that $L$ is not $\mathcal{N}_{\alpha+1}$-complete.

Suppose $L$ is $\mathcal{N}_{\alpha+1}$-complete. Then, since $L = [0, X \cup Y]$, by the additivity of $\mathcal{N}_{\alpha+1}$-continuity in $\mathcal{N}_{\alpha+1}$-complete lattices, $L$ is $\mathcal{N}_{\alpha+1}$-continuous. Let $\Omega$ be the first ordinal such that $\overline{\Omega} = \mathcal{N}_{\alpha+1}$ and $\{a^\beta\}_{\beta < \Omega}$ an increasing chain of elements of $\mathcal{N}$ such that $\bigcup a^\beta \notin \mathcal{N}$. Take $C^\beta = \begin{pmatrix} 0 & 0 \\ a^\beta & a^\beta \end{pmatrix}$

Then $C = \bigcup C^\beta = \begin{pmatrix} 0 & 0 \\ \bigcup a^\beta & \bigcup a^\beta \end{pmatrix} \notin L$.

If $C' = \begin{pmatrix} e_1 & 0 \\ b & * \end{pmatrix}$ is the supremum of the $C^\beta$ in $L$ then $b \notin \bigcup a^\beta$, since $b \in I$. On the other hand $C < C'$ implies that $\bigcup a^\beta \leq b \cup e_1$, hence $e_1 \notin 0$. Now $D = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \in L$. $D \cap C^\beta = 0$ for all $\beta < \Omega$, but $D \cap C \neq 0$, which contradicts the $\mathcal{N}_{\alpha+1}$-continuity of $L$.

COROLLARY 2. Let $L$ be as in Theorem 1. Then $L$ is a Von Neumann geometry if and only if $\mathcal{G}$ is a complete Boolean algebra and $\mathcal{I}$ is a principal ideal, that is, $I = [0, x]$, $x \in B$. In this case the center of $L$ is isomorphic to $[0, x] \times [0, x'] \times [0, x']$. 

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Proof. When \( \mathcal{I} = [0, x] \), \( L \) is the lattice direct sum of the intervals \([0, Y_0], [0, Y_1], [0, Y_2]\), where

\[
Y_0 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad Y_1 = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & x' \end{pmatrix}.
\]

Hence its center is isomorphic to \([0, x] \times [0, x] \times [0, x]\).

**Theorem 2.** If \( \mathcal{B} \) is a complete Boolean algebra, then the lattice \( L_B^n \) is a Von Neumann geometry whose center is isomorphic to \( \mathcal{B} \).

Remark. For \( n = 2 \) this theorem is contained in Theorem 1.

Proof. Because of the dual isomorphism between \( L_B^n \) and \( \mathcal{B} \), we only need to prove that \( L_B^n \) is upper complete and upper continuous. Now \( L_B^n = [0, X_1 \cup X_2 \cup \ldots \cup X_n] \), where \( X_i \) is the canonical matrix with 1 in the \((i, i)\) place and zeros elsewhere, and the interval \([0, X_i]\) being isomorphic to \( \mathcal{B} \), is continuous. Therefore, by the theorem of Amemiya and Halperin quoted in the introduction, if \( L_B^n \) is upper complete it is also upper continuous. So it is sufficient to prove that \( L_B^n \) is upper complete.

We use induction on \( n \). If \( n = 1 \), \( L_B \cong \mathcal{B} \) and there is nothing to prove. Assume then that the theorem is true for \( n-1 \). Let \( \Lambda^\beta \) be an increasing chain, where \( \beta < \Omega \), \( \Omega \) any limit ordinal, and

\[
E = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \in L_B^n.
\]
Then the elements $A^\beta \cap E$ form an increasing chain. To the element $A^\beta \cap E$ there corresponds a finitely generated submodule $N^\beta$ of the left $B$-module of $n$-tuples and the elements of this submodule have the last component equal zero. Therefore, because of the induction assumption, the increasing chain of submodules $N^\beta$ has a supremum which is also a submodule whose elements have the last component equal to zero. Let $A' \in \mathbb{L}_B^n$ be the left canonical matrix corresponding to this submodule,

$$
\begin{pmatrix}
e_1 & 0 & \cdots & 0 & 0 \\
e_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
$$

If $C$ is an upper bound of the $A^\beta$, $\beta \in \Omega$, then $C \geq A^\beta \cap E$. Hence $C \geq A'$, and $C \geq A^\beta \cup A'$. That is, any upper bound of the $A^\beta$ is an upper bound of the chain of $A^\beta \cup A'$ and conversely. Let $B^\beta = A^\beta \cup A'$, since $B^\beta \cap E = (A^\beta \cup A') \cap E = A'$,

$$
\begin{pmatrix}
e_1 & 0 & \cdots & 0 & 0 \\
e_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
$$

Moreover, if $\alpha < \beta$, $B^\alpha \subset B^\beta$ and this implies $B^\alpha B^\beta = B^\alpha$, which is equivalent to $b_i^\alpha b_i^\beta = b_i^\alpha$, $i = 1, 2, \ldots, n-1$, $e_n^\alpha e_n^\beta = e_n^\alpha$. Now it is easily seen that
\[
B = \begin{pmatrix}
e_1 & 0 & \cdots & 0 & 0 \\
e_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\cup b_1^\beta & \cup b_2^\beta & \cdots & \cup b_{n-1}^\beta & \cup e_n^\beta
\end{pmatrix}
\]

is the supremum of the chain of \( B^\alpha \). For, \( e_n^\alpha b_i^\beta = b_i^\alpha \) and \( e_n^\alpha e_n^\beta = e_n^\alpha \) for \( \alpha < \beta \) imply that the \( b_i^\beta \) and \( e_n^\beta \) form increasing chains. Consequently, \( e_n^\alpha (\cup b_i^\beta) = b_i^\alpha \), \( e_n^\alpha (\cup e_n^\beta) = e_n^\alpha \) and \( (\cup e_n^\alpha) (\cup b_i^\beta) = \cup (e_n^\alpha (\cup b_i^\beta)) = \cup b_i^\alpha \).

Therefore \( B \) is a canonical matrix such that \( B^\alpha B = B^\alpha \), which implies \( B^\alpha \leq B \), and it is clear that \( B \) is the supremum.

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